Solution of the KdV equation using evolutionary vessels

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November 10, 2011

Abstract

In this work we present a new method for solving of the Korteweg-de Vries (KdV) equation

$$q'_t = -\frac{3}{2}qq'_x + \frac{1}{4}q'''_{xxx}.$$

The proposed method is a particular case of the theory of evolutionary vessels, developed in this work. Inverse scattering of the Sturm-Liouville operator and evolution of its potential are the basic ingredients, similar to the existing methods developed by Gardner-Greene-Kruskal-Miura (1967), Zacharov-Shabbath (1974) and Peter Lax (1977). Evolutionary KdV vessel may be considered as a generalization of these previous works. The advantage of the new method is that it produces a unified approach to existing solutions of the KdV equation. For example, odd or even analytic, periodic, almost periodic solutions are shown to be particular cases of this theory. Generalizing this method we can also produce many PDEs, associated with integrable systems, in an arbitrary number of variables (in the spirit of Zakarov-Shabat).

1 Introduction

The Korteweg-de Vries (KdV) is the following nonlinear evolutionary PDE

$$q'_t = -\frac{3}{2}qq'_x + \frac{1}{4}q'''_{xxx}. \tag{1}$$

The standard technique for solving this equation [CG67] consists of the detailed study of the evolution of the scattering data corresponding to the potential $q(x,t)$ of Sturm-Liouville differential equation

$$-y''_{xx} + q(x,t)y = \lambda y. \tag{2}$$

Zakarov-Shabat [ZABŠ74] have created a similar scheme for a wide family of nonlinear equations using the ideas of P. Lax [Lax77], or more precisely a Lax pair. We will present an alternative scheme, similar to these ideas, for constructing solutions of the KdV equation (1).

The starting point is a 2×2 matrix-function of a complex variable λ, realized [BGR90] in the following form using an auxiliary Hilbert space $\mathcal{H}$:

$$S(\lambda) = I - B_0^* \mathbb{X}_0^{-1}(\lambda I - A)^{-1} B_0 \sigma_1,$$
$$A\mathbb{X}_0 + \mathbb{X}_0 + A^* + B_0\sigma_1 B_0^* = 0, \quad \mathbb{X}_0^* = \mathbb{X}_0,$$

where $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B_0 : \mathbb{C}^2 \to \mathcal{H}$, $A, \mathbb{X}_0 : \mathcal{H} \to \mathcal{H}$ are linear operators. These kinds of realizations were studied for example in [BGR90, MB58, Mel09, Mela, Melb, Pot55, Sc68, DAdS97, Bi71]. The main

feature in the study of such realizations is the correspondence between the multiplicative structure of the matrix $S(\lambda)$ and the invariant subspaces of the operator A [MB58, Pot55]. The pioneering work [Ls78] of M. Livšič and its generalization in [Ls01] are the main sources of motivation for this research project.

The variable λ may be called the "frequency" variable. Define next $\sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\gamma = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$ and solve the following differential equations with initial conditions in the order they appear, starting from $B_0, \mathbb{X}_0$ appearing in the realization of $S(\lambda)$ above:

$$0 = \tfrac{d}{dx}(B(x)\sigma_1) + AB(x)\sigma_2 + B(x)\gamma \qquad\qquad B(0) = B_0, \qquad (5)$$
$$\tfrac{d}{dx}\mathbb{X}(x) = B(x)\sigma_2 B^*(x), \qquad\qquad \mathbb{X}(0) = \mathbb{X}_0, \qquad (7)$$
$$\frac{\partial}{\partial t}B(x,t) = iA\frac{\partial}{\partial x}B(x,t), \qquad\qquad B(x,0) = B(x), \quad (22)$$
$$\frac{\partial}{\partial t}\mathbb{X}(x,t) = iAB(x,t)\sigma_2 B^*(x,t) - iB(x,t)\sigma_2 B^*(x,t)A^* + iB(x,t)\gamma B^*(x,t), \quad \mathbb{X}(x,0) = \mathbb{X}(x). \quad (23)$$

Then we can check that the Lyapunov equation

$$A\,\mathbb{X}(x,t) + \mathbb{X}(x,t)A^* + B(x,t)\sigma_1 B(x,t) = 0$$

holds for all x, t (the derivative with respect to x and t is zero). The so-called "tau function" of this construction is

$$\tau(x,t) = \det(\mathbb{X}_0^{-1}\mathbb{X}(x,t))$$

and has the property that $q(x,t) = -2\dfrac{\partial^2}{\partial x^2}\tau(x,t)$ satisfies the KdV equation (1) (Theorem 10). In Theorem 9 it is shown that when the function $S(\lambda)$ is meromorphic (i.e. has a finite-dimensional realization, $\mathcal{H} < \infty$), we obtain a soliton. The collection of these constructed operators and spaces

$$\mathfrak{E}_{KdV} = (A, B(x,t), \mathbb{X}(x,t); \sigma_1, \sigma_2, \gamma, \gamma_*(x,t); \mathcal{H}, \mathbb{C}^2).$$

is called a KdV evolutionary vessel (see Definition 8).

In this work we show that the scheme proposed here for constructing solutions of the KdV equation is effective and that some important solutions such as solitons, periodic, almost periodic, and analytic are obtained in this way. Moreover, for the case of odd/even in x, t solutions of (1) we show in Section 4 that there is also a 1-1 correspondence with some "canonical" constructions of the KdV vessels $\mathfrak{E}_{KdV}$.

Section 2 is devoted to the vessel, obtained using only the first two steps (5) and (7), and as a result it deals with operators depending on x only. These types of vessels were studied by the author and others in a series of papers [Mel09, Mela, MVb, Melb] and they represent a basis for this theory. The definition and basic ideas of vessels follow from the work [Ls78] of M. Livšič. In Section 2.4 it is shown that the classical inverse scattering fits into this setting and that the remaining two equations (22), (23) are equivalent to the evolutions of the potential $q(x,t)$, or of its scattering data as in the classical case.

In Section 4 we prove that there is 1-1 correspondence between certain odd at $t = 0$ classes of solutions and vessels. These correspondences are presented in Table 1.

It seems, and is a project by itself, that the solution of G. Segal, G. Wilson using Loop groups [GS85] corresponds to some choices of the initial function $S(\lambda)$. The Krichever solutions [Kri77], which can also be implemented using loop groups [GS85], are conjecturally examples of vessels implemented on curves [Melb]. Baker functions [Bak28] seem to be generalized by the transfer function of the vessel

$$S(\lambda, x, t) = I - B^*(x,t)\mathbb{X}^{-1}(x,t)(\lambda I - A)^{-1}B(x,t)\sigma_1.$$

Finally, the Zakarov-Shabat scheme [ZABŠ74] can be generalized by adding variables to the operator $B, \mathbb{X}$ and defining linear differential equations in the spirit of (5), (7), (22), and (23). For example, we should be able to produce solutions of the Kadomtzev-Petviashvili equation [BK70] using this scheme.

Class of solutions	Type of evol. vessel	Previous works	Correspondence
Solitons	$\dim \mathcal{H} < \infty$	[BB00]	Theorem 9
Analytic-exponential	Regular vessels	[RB88]	Theorem 16
$\exists q'''_{xxx}(x,t)$ on $\mathbb{R}$	$\operatorname{spec}(A) \subseteq i\mathbb{R}_+$	[BB00, Lax77], [ZABŠ74, CG67]	Theorem 25
T-periodic in x $\dfrac{T^3}{(2\pi)^2}$-periodic in t	$\mathcal{H} = \ell^2,\ A = \operatorname{diag}(\sqrt{-1}n_i^2),$ $n_i = \dfrac{2\pi N_i}{T}, N_i \in \mathbb{N}$	[Lax75], [Kri77]	Theorem 29
almost periodic	$\mathcal{H} = \ell^2,\ A = \operatorname{diag}(\sqrt{-1}r_i^2),$ $r_i \in \mathbb{R}$	[Lax76]	Theorem 30

Table 1: Classes of KdV odd solutions and corresponding vessels.

2 Sturm-Liouville vessels

2.1 Sturm-Liouville Vessels

Definition 1. *The Sturm Liouville (SL) vessel parameters are defined as follows*

$$\sigma_1 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \sigma_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \gamma = \left[\begin{array}{cc} 0 & 0 \\ 0 & i \end{array} \right], \gamma_*(x) = \left[\begin{array}{cc} -i(\beta'(x) - \beta^2(x)) & -\beta(x) \\ \beta(x) & i \end{array} \right]$$

for a real-valued differentiable function $\beta(x)$, defined on an interval I.

Before we define a notion of a vessel which involves an auxiliary Hilbert space $\mathcal{H}$ and operators (for $x \in$ I)

$$\begin{array}{ccccc} A, \mathbb{X}(x) & : & \mathcal{H} & \to & \mathcal{H}, \\ B(x) & : & \mathbb{C}^2 & \to & \mathcal{H} \end{array} \tag{3}$$

we need to consider some regularity assumptions. We assume that the operator A may be unbounded with a domain $D(A)$. Moreover, certain algebraic and differential relations connect these operators. As as a result, we must determine assumptions, which ensure that the relations between $A, \mathbb{X}(x), B(x)$ become solvable equations.

Definition 2 (Regularity assumptions). *Operators $A, \mathbb{X}(x), B(x)$ are said to satisfy regularity assumptions on* I *if*

1. *A is a generator of C_0 semi-group,*

2. *$B(x)\mathbb{C}^2 \in D(A^n)$ for all $x \in$ I$, n \in \mathbb{N}$,*

3. *The operator $\mathbb{X}(x)$ is self-adjoint and invertible for all $x \in$ I.*

Definition 3. *A Strum-Liouville vessel is a collection of operators and spaces*

$$\mathfrak{V} = (A, B(x), \mathbb{X}(x); \sigma_1, \sigma_2, \gamma, \gamma_*(x); \mathcal{H}, \mathbb{C}^2), \tag{4}$$

The space $\mathcal{H}$ is Hilbert and the operators $A, \mathbb{X}(x), B(x)$ are defined in (3) so that the regularity assumptions

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*hold. The operators are subject to the following **vessel conditions**:*

$$0 = \frac{\partial}{\partial x}(B\sigma_1) + AB\sigma_2 + B\gamma, \tag{5}$$

$$A\mathbb{X} + \mathbb{X}A^* = B\sigma_1 B^*, \tag{6}$$

$$\frac{\partial}{\partial x}\mathbb{X} = B\sigma_2 B^*, \tag{7}$$

$$\gamma_* = \gamma + \sigma_2 B^* \mathbb{X}^{-1} B\sigma_1 - \sigma_1 B^* \mathbb{X}^{-1} B\sigma_2. \tag{8}$$

By definition, the **transfer function** of this vessel is

$$S(\lambda, x) = I - B^*(x)\mathbb{X}^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1.$$

Poles and singularities of S with respect to λ are determined by A only. Multiplication by the function $S(\lambda, x)$ maps [Mel09, Melb, MVa] solutions $u(\lambda, x)$ of the input LDE with the spectral parameter λ

$$\lambda \sigma_2 u(\lambda, x) - \sigma_1 \frac{\partial}{\partial x} u(\lambda, x) + \gamma u(\lambda, x) = 0 \tag{9}$$

to solutions $y(\lambda, x) = S(\lambda, x)u(\lambda, x)$ of the output LDE with the same spectral parameter

$$\lambda \sigma_2 y(\lambda, x) - \sigma_1 \frac{\partial}{\partial x} y(\lambda, x) + \gamma_*(x)y(\lambda, x) = 0. \tag{10}$$

Denoting $u_\lambda(x) = \begin{bmatrix} u_1(\lambda, x) \\ u_2(\lambda, x) \end{bmatrix}$ we shall obtain that the input compatibility condition (9) is equivalent to

$$\begin{cases} -\frac{\partial^2}{\partial x^2} u_1(\lambda, x) = -i\lambda u_1(\lambda, x), \\ u_2(\lambda, x) = -i\frac{\partial}{\partial x} u_1(\lambda, x). \end{cases}$$

The output $y_\lambda(x) = \begin{bmatrix} y_1(\lambda, x) \\ y_2(\lambda, x) \end{bmatrix} = S(\lambda, x)u_\lambda(x)$ satisfies the output equation (10), which is equivalent to

$$\begin{cases} -\frac{\partial^2}{\partial x^2} y_1(\lambda, x) + 2\beta'(x)y_1(\lambda, x) = -i\lambda y_1(\lambda, x), \\ y_2(\lambda, x) = -i[\frac{\partial}{\partial x} y_1(\lambda, x) - \beta(x)y_1(\lambda, x)]. \end{cases}$$

In other words, we can see that multiplication by $S(\lambda, x)$ maps the solution of the trivial SL equation (2) ($q = 0$) to solutions of the more complicated one, defined by the potential

$$q(x) = 2\beta'(x). \tag{11}$$

As a result, the function $S(\lambda, x)$ satisfies the following differential equation

$$\frac{\partial}{\partial x}S = \sigma_1^{-1}(\lambda \sigma_2 + \gamma_*)S - S\sigma_1^{-1}(\lambda \sigma_2 + \gamma). \tag{12}$$

We notice that $(\mathcal{H}, \mathbb{X}(x))$ form a Krein space: $\mathcal{H}$ is a set, whose (Krein) inner product $[u, v] = \langle \mathbb{X}(x)u, v\rangle, \forall u, v \in \mathcal{H}$ depends on x and is differentiable. From system theory [Bi71, KR69] and operator theory related to J-contractive functions [BGR90, Pot55] we adopt some of the following additional characterizations of the vessel.

Definition 4. *The vessel $\mathfrak{V}$ (4) is called*

- *__dissipative__, if $\mathbb{X}(x) > 0$ for all values of $x \in \mathrm{I}$,*
- *__Pontryagin__, if $\mathbb{X}(x)$ has $\kappa \in \mathbb{N}$ negative squares at the right half plane for all values of $x \in \mathrm{I}$,*
- *__Krein__, if $\mathbb{X}(x)$ has infinite number of negative squares at the right half plane for all values of $x \in \mathrm{I}$,*
- *__regular__, if all the operators $A, B(x), \mathbb{X}(x)$ are bounded operators for all x,*
- *__minimal__, if for all x,*

$$\overline{\mathrm{span}}\{A^n B(x)\mathcal{E} \mid n \in \mathbb{N}\} = \mathcal{H}. \tag{13}$$

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2.2 Standard construction of a vessel

in this section we show that one can easily construct vessels. Choose two Hilbert spaces $\mathcal{H}, \mathcal{E}$ and define three operators $\mathbb{X}_0, A : \mathcal{H} \to \mathcal{H}$ and $B_0 : \mathcal{E} \to \mathcal{H}$ such that $\mathbb{X}_0$ is invertible and the following equalities hold:
$$\mathbb{X}_0^* = \mathbb{X}_0, \quad A\mathbb{X}_0 + \mathbb{X}_0 A^* + B_0 \sigma_1 B_0^* = 0.$$

Then solve (5) with the initial value
$$0 = \frac{d}{dx}(B(x)\sigma_1) + AB(x)\sigma_2 + B(x)\gamma, \quad B(x_0) = B_0,$$

and solve equation (7):
$$\mathbb{X}(x) = \mathbb{X}_0 + \int_{x_0}^{x} B(y)\sigma_2 B(y)^* dy.$$

Finally, define $\gamma_*(x)$ from $\gamma(x)$ using (8). Thus a vessel is created [Melb]:

Lemma 5. *The collection*
$$\mathfrak{K}_{\mathfrak{V}} = (A, B(x), \mathbb{X}(x); \sigma_1, \sigma_2, \gamma, \gamma_*(x); \mathcal{H}, \mathcal{E}; \mathrm{I})$$

is a vessel.

Proof: Follows from the definitions. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We can obtain in this manner a rich family of vessels, using standard models, which create operators $\mathbb{X}_0, A, B_0$:

1. Livšic model of a non-selfadjoint operator [MB58], where $\mathbb{X} = I$, $A + A^* + BJB^* = 0$, and J is a signature matrix $(J = J^* = J^{-1})$,

2. Theory of nodes, developed in [Bi71],

3. Krein space realizations for symmetric functions [AD],

4. Vessels on curves [Melb],

The most important aspect of all these constructions is that the singularities of the transfer function are manageable. Under the minimality condition (see Definition 4), taking the operator A as a diagonal operator, we obtain that these singularities are at certain points $(\dim \mathcal{H} < \infty)$ or on a countable set of points $(\mathcal{H} = \ell^2)$ or on a curve Γ $(\mathcal{H} = L^2(\Gamma))$. In this manner we can construct vessels whose transfer function will have singularities at the prescribed points (or on prescribed curves). The converse of this construction is an analogue of inverse scattering.

2.3 The tau function of a vessel

Following the ideas presented in [Mela], we define the tau function of the vessel $\mathfrak{V}$, (see (4)) as follows

Definition 6. *For a given vessel*
$$\mathfrak{V} = (A, B(x), \mathbb{X}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x); \mathcal{H}, \mathcal{E}; \mathrm{I}),$$

the tau function $\tau(x)$ is defined as
$$\tau = \det(\mathbb{X}^{-1}(x_0)\mathbb{X}(x)) \tag{14}$$

for an arbitrary point $x_0 \in \mathrm{I}$.

Notice that, using vessel condition (7), $\mathbb{X}(x)$ has the formula

$$\mathbb{X}(x) = \mathbb{X}(x_0) + \int\limits_{x_0}^{x} B^*(y)\sigma_2 B(y)dy,$$

and as a result

$$\mathbb{X}^{-1}(x_0)\mathbb{X}(x) = I + \mathbb{X}^{-1}(x_0)\int\limits_{x_0}^{x} B^*(y)\sigma_2 B(y)dy.$$

Since σ_2 has rank 1, this expression is of the form $I + T$, for a trace class operator T. Since $\mathbb{X}_0$ is an invertible operator, there exists a non-trivial interval (of length at least $\dfrac{1}{\|\mathbb{X}_0^{-1}\|}$) on which $\mathbb{X}(x)$ and $\tau(x)$ are defined. Recall [IG69] that a function $F(x)$ from (a, b) into the group G (the set of bounded invertible operators on H of the form $F(x) = I + T(x)$) is said to be differentiable if $F(x) - I$ is *differentiable* as a map into the trace-class operators. In our case,

$$\frac{d}{dx}(\mathbb{X}^{-1}(x_0)\mathbb{X}(x)) = \mathbb{X}^{-1}(x_0)\frac{d}{dx}\mathbb{X}(x) = \mathbb{X}^{-1}(x_0)B(x)\sigma_2 B^*(x)$$

exists in trace-class norm. Israel Gohberg and Mark Krein [IG69, formula 1.14 on p. 163] proved that if $\mathbb{X}^{-1}(x_0)\mathbb{X}(x)$ is a differentiable function into G, then $\tau(x) = \mathrm{sp}(\mathbb{X}^{-1}(x_0)\mathbb{X}(x))$ [1] is a differentiable map into $\mathbb{C}^*$ with

$$\frac{\tau'}{\tau} = \mathrm{sp}\left(\left(\mathbb{X}^{-1}(x_0)\mathbb{X}(x)\right)^{-1}\frac{d}{dx}\left(\mathbb{X}^{-1}(x_0)\mathbb{X}(x)\right)\right) = \mathrm{sp}(\mathbb{X}(x)'\mathbb{X}^{-1}(x))$$

$$= \mathrm{sp}(B(x)\sigma_2 B^*(x)\mathbb{X}^{-1}(x)) = \mathrm{tr}(\sigma_2 B^*(x)\mathbb{X}^{-1}(x)B(x)). \quad (15)$$

Since any two realizations of a symmetric function are (weakly) isomorphic, one obtains from standard theorems [BGR90] in realization theory of analytic at infinity functions that they will have the same tau function up to a scalar. In other words, this notion is independent of the realization we choose for the given function $S(\lambda, x)$.

2.4   Scattering data versus $S(\lambda, 0)$ and Gelfand-Levitan equation

Following [Fad63] for the case $\int\limits_0^\infty x|q(x)|dx < \infty$, there are introduced Jost solutions $\phi(x,s)$ and $f(x,s)$ [Jos47, Lev49]

$$\phi(x,s) \quad : \quad \phi(0,s) = 0, \quad \phi'(0,s) = 1, \tag{16}$$

$$f(x,s) \quad : \quad \lim_{x\to\infty} e^{-isx}f(x,s) = 1. \tag{17}$$

Defining further $M(k) = \phi'(x,k)f(x,s) - f'(x,s)\phi(x,k)$ one reconstructs the potential $q(x)$ using the Gelfand-Levitan equation [IMG51] (or alternatively the Marchenko equation [Mc50]). There are two steps essential for this construction: one considers the case when the spectrum of the operator L is purely continuous and the case when this spectrum additionally contains finite number of points. For the purely continuous case, one proves that there is a solution $K(x,y)$ of the Gelfand-Levitan equation [Fad63, (8.5)]

$$K(x,y) + \Omega(x,y) + \int\limits_0^x K(x,t)\Omega(t,y)dt = 0, \quad x > y, \tag{18}$$

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[1]sp - stands for the trace in the infinite dimensional space.

where $\Omega(x, y)$ is uniquely defined from $M(k)$ by [Fad63, (8.4)]

$$\Omega(x, y) = 2/\pi \int_0^\infty \frac{\sin(kx)}{k} \left[\frac{1}{M(k)M(-k)} - 1\right] \frac{\sin(ky)}{k} k^2 dk.$$

The formula for the potential is [Fad63, (10.4)] $q(x) = 2\frac{d}{dx} K(x, x)$. Then one make a modification, so that the discrete spectrum is taken into account [Fad63, (8.14, 8.15)]. We will present analogues of these formulas in our setting. Suppose that we are given an SL vessel (4)

$$\mathfrak{V} = (A, B(x), \mathbb{X}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x); \mathcal{H}, \mathcal{E}; \mathrm{I}),$$

and let us fix an arbitrary $x_0 \in \mathrm{I}$. Define

$$\Omega(x, y) = \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}(x_0)B(y)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{19}$$

and

$$K(x, y) = -\begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}(x)B(y)\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{20}$$

Then the Gel'fand-Levitan equation (18) holds

$K(x, y) + \Omega(x, y) + \int_{x_0}^x K(x, t)\Omega(t, y)dt =$

$= K(x, y) + \Omega(x, y) - \int_{x_0}^x \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}(x)B(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} B^*(t)\mathbb{X}^{-1}(x_0)B(y)\begin{bmatrix} 1 \\ 0 \end{bmatrix} dt$

$= K(x, y) + \Omega(x, y) - \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}(x)\int_{x_0}^x B(t)\sigma_2 B^*(t)dt \mathbb{X}^{-1}(x_0)B(y)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$=$ using vessel condition (5)

$= K(x, y) + \Omega(x, y) - \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}(x)(\mathbb{X}(x) - \mathbb{X}(x_0))\mathbb{X}^{-1}(x_0)B(y)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= K(x, y) + \Omega(x, y) - \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}(x_0)B(y)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}(x)B(y)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$

Finally, the formula (11) for the potential gives

$q(x) = 2\frac{d}{dx}\mathrm{sp}(\mathbb{X}^{-1}(x)\frac{d}{dx}\mathbb{X}(x)) = \mathrm{sp}(\mathbb{X}^{-1}(x)B(x)\sigma_2 B^*(x)) =$

$= 2\frac{d}{dx}\mathrm{sp}(\mathbb{X}^{-1}(x)B(x)\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)) = \frac{d}{dx}(\begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)\mathbb{X}^{-1}(x)B(x)\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \tag{21}$

$= -2\frac{d}{dx}K(x, x),$

which is identical to [Fad63, (10.4)].

3   KdV evolutionary vessels

To give motivation for the definition of a KdV evolutionary vessel, we present a realization of the 3-soliton [BB00, Example 3.3]. Choose $\mathcal{H} = \mathbb{C}^3$ and consider the collection, defining a vessel, where the operators $B, \mathbb{X}$ depend on the variable t:

$$\mathfrak{E}_{KdV} = (A, B(x, t), \mathbb{X}(x, t); \sigma_1, \sigma_2, \gamma, \gamma_*(x, t); \mathbb{C}^3, \mathbb{C}^2).$$

Suppose that the function $B(x,t)$ satisfies, in addition to (5), the following differential equation

$$\frac{\partial}{\partial t} B(x,t) = iA \frac{\partial}{\partial x} B(x,t) \tag{22}$$

and that $\mathbb{X}(x,t)$ satisfies

$$\frac{\partial}{\partial t} \mathbb{X} = iAB\sigma_2 B^* - iB\sigma_2 B^* A^* + iB\gamma B^*. \tag{23}$$

Then for some nonzero constants b_1, b_2, b_3 and mutually different positive k_i's we can solve all these equations to obtain the following formulas:

$$A = \operatorname{diag}(-ik_j^2) = \begin{bmatrix} -ik_1^2 & 0 & 0 \\ 0 & -ik_2^2 & 0 \\ 0 & 0 & -ik_3^2 \end{bmatrix} = -A^*,$$

$$B(x,t) = \begin{bmatrix} e^{k_1 x + k_1^3 t} b_1 & 0 & 0 \\ 0 & e^{k_2 x + k_2^3 t} b_2 & 0 \\ 0 & 0 & e^{k_3 x + k_3^3 t} b_3 \end{bmatrix} \begin{bmatrix} 1 & ik_1 \\ 1 & ik_2 \\ 1 & ik_3 \end{bmatrix},$$

$$\mathbb{X}(x,t) = I + [\frac{e^{(k_i+k_j)x + (k_i^3+k_j^3)t}}{k_i+k_j} b_i b_j^*].$$

The main reason why this example is interesting is the following lemma.

Lemma 7. *The tau function of the vessel $\mathfrak{V}_{KdV}$ is*

$$\tau(x,t) = \det \mathbb{X}(x,t) = 1 + \sum_{i=1}^{3} c_i e^{2k_i x + 2k_i^3 t}$$

$$+ \sum_{1 \le i < j \le 3} c_i c_j a_{ij} e^{2(k_i+k_j)x + 2(k_i^3+k_j^3)t}$$

$$+ c_1 c_2 c_3 a_{12} a_{13} a_{23} e^{2(k_1+k_2+k_3)x + 2(k_1^3+k_2^3+k_3^3)t},$$

where $a_{ij} = \dfrac{(k_i-k_j)^2}{(k_i+k_j)^2}$ and $c_i = \dfrac{|b_i|^2}{2k_i}$.

Proof: Calculate the determinant, using the formula for Cauchy determinant. $\qquad\square$

Notice that the above formula coincides with the example of 3-soliton in [BB00, Example 3.3].

Definition 8. *A **KdV (evolutionary) vessel** is a collection of operators and spaces*

$$\mathfrak{V}_{KdV} = (A, B(x,t), \mathbb{X}(x,t); \sigma_1, \sigma_2, \gamma, \gamma_*(x); \mathcal{H}, \mathcal{E}),$$

where $\sigma_1, \sigma_2, \gamma$ are SL parameters (see Definition 1). The operators are subject to the vessel conditions (5), (6), (7), (8), (22), and (23) and the normalization condition

$$\operatorname{tr}(\sigma_1 B^* \mathbb{X}^{-1} B) = 0. \tag{24}$$

The normalization condition becomes much simpler in certain cases:

$$\operatorname{tr}(\sigma_1 B^* \mathbb{X}^{-1} B) = \operatorname{tr}(B\sigma_1 B^* \mathbb{X}^{-1}) = \operatorname{tr}((A\mathbb{X} + \mathbb{X}A^*)X^{-1}) = \operatorname{tr}(A + A^*),$$

if the last expression make sense. Here is an example of such a vessel, that implements solitons.

Theorem 9. *Let $\sigma_1, \sigma_2, \gamma$ be defined as in Definition 1. Let k_i $(i = 1, \ldots, n)$ be mutually different positive real numbers, and $b_i \in \mathbb{C}$ $(i = 1, \ldots, n)$ nonzero. Define the collection*

$$\mathfrak{V}_{KdV} = (A, B(x,t), \mathbb{X}(x,t); \sigma_1, \sigma_2, \gamma, \gamma_*(x); \mathbb{C}^n, \mathcal{E}),$$

where

$$
\begin{aligned}
A \quad &= \operatorname{diag}(-ik_j^2) = -A^*, \\
B(x,t) \quad &= \operatorname{diag}(e^{k_j x + k_j^3 t} b_j) \begin{bmatrix} 1 & ik_1 \\ \vdots & \vdots \\ 1 & ik_n \end{bmatrix}, \\
\mathbb{X}(x,t) \quad &= I_n + \left[\frac{e^{(k_i+k_j)x + (k_i^3 + k_j^3)t}}{k_i + k_j} b_i b_j^* \right]
\end{aligned}
\tag{25}
$$

and γ_* is defined using (8). Then $\mathfrak{V}_{KdV}$ is a KdV evolutionary vessel. The tau function of this vessel $\tau(x,t) = \det(\mathbb{X}(x,t))$ is the tau function of an n-soliton of the KdV equation.

Proof: Let us show that the equations of the vessel hold. Equation (5) is derived from the following calculations:

$$\frac{\partial}{\partial x} B(x,t) = \operatorname{diag}(k_j) B(x,t),$$

$$-(AB(x,t)\sigma_2 + B(x,t)\gamma)\sigma_1^{-1} = \operatorname{diag}(ik_j^2) B(x,t) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - B(x,t) \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}$$

$$= \operatorname{diag}(ik_j^2) \operatorname{diag}(e^{k_j x + k_j^3 t} b_j) \left[\begin{bmatrix} 1 & ik_1 \\ \vdots & \vdots \\ 1 & ik_n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \operatorname{diag}(e^{k_j x + k_j^3 t} b_j) \begin{bmatrix} 1 & ik_1 \\ \vdots & \vdots \\ 1 & ik_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix} \right] =$$

$$= \operatorname{diag}(e^{k_j x + k_j^3 t} b_j) \left[\begin{bmatrix} 0 & ik_1^2 \\ \vdots & \vdots \\ 0 & ik_n^2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ \vdots & \vdots \\ k_n & 0 \end{bmatrix} \right] = \operatorname{diag}(e^{k_j x + k_j^3 t} b_j) \left[\begin{bmatrix} k_1 & ik_1^2 \\ \vdots & \vdots \\ k_n & ik_n^2 \end{bmatrix} \right] =$$

$$= \operatorname{diag}(k_j) B(x,t).$$

The Lyapunov equation (6) is derived as follows, where the first equality comes from the fact that $A = -A^*$:

$$A\mathbb{X}(x,t) + \mathbb{X}(x,t)A^* + B(x,t)\sigma_1 B^*(x,t) =$$

$$= A\left(I + \left[\frac{e^{(k_i+k_j)x + (k_i^3 + k_j^3)t}}{k_i + k_j} b_i b_j^*\right]\right) + \left(I + \left[\frac{e^{(k_i+k_j)x + (k_i^3 + k_j^3)t}}{k_i + k_j} b_i b_j^*\right]\right)(-A) + B(x,t)\sigma_1 B^*(x,t)$$

$$= \operatorname{diag}(-ik_j^2)\left[\frac{e^{(k_i+k_j)x + (k_i^3 + k_j^3)t}}{k_i + k_j} b_i b_j^*\right] + \left[\frac{e^{(k_i+k_j)x + (k_i^3 + k_j^3)t}}{k_i + k_j} b_i b_j^*\right] \operatorname{diag}(ik_j^2)$$

$$+ \operatorname{diag}(e^{k_j x + k_j^3 t} b_j) \begin{bmatrix} 1 & ik_1 \\ \vdots & \vdots \\ 1 & ik_n \end{bmatrix} \sigma_1 \begin{bmatrix} 1 & \cdots & 1 \\ -ik_1 & \cdots & -ik_n \end{bmatrix} \operatorname{diag}(e^{k_j x + k_j^3 t} b_j)$$

$$= \left[\frac{e^{(k_i+k_j)x + (k_i^3 + k_j^3)t}}{k_i + k_j} b_i b_j^* (\sqrt{-1}(-k_i^2 + k_j^2))\right] + \left[e^{(k_i+k_j)x + (k_i^3 + k_j^3)t} b_i b_j^* \sqrt{-1}(k_i - k_j)\right]$$

$$= \left[e^{(k_i+k_j)x + (k_i^3 + k_j^3)t} b_i b_j^* \left(\frac{\sqrt{-1}(-k_i^2 + k_j^2)}{k_i + k_j} + \sqrt{-1}(k_i - k_j)\right)\right] = 0.$$

Equations (7) and (22) are immediate. Finally, the fact that the tau function is the tau function of an n-soliton follows from the more general Theorem 10. The normalization condition (24) is also immediate in this case. □

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Theorem 10. *Let $\mathfrak{V}_{KdV}$ be a KdV vessel, then the potential $q(x,t)$ of the output SL equation (10) satisfies KdV equation (1):*

$$q'_t = -\frac{3}{2}qq'_x + \frac{1}{4}q'''_{xxx}.$$

Proof: By (11), the KdV equation for $q(x,t)$ follows from the differential equation for $\beta(t,x)$:

$$4\beta'_t = -6(\beta'_x)^2 + \beta'''_{xxx}. \tag{26}$$

Starting from $\dfrac{\tau'}{\tau} = -\beta$ and (15), we obtain that

$$\beta(t,x) = -\begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x,t)\mathbb{X}^{-1}(x,t)B(x,t)\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The next formula follows from (5), (6), and (7):

$$\sigma_1(B^*\mathbb{X}^{-1})' = \sigma_2 B^*\mathbb{X}^{-1}A + \gamma_* B^*\mathbb{X}^{-1},$$

for $\gamma_* = \begin{bmatrix} -i(\beta' - \beta^2) & -\beta \\ \beta & i \end{bmatrix}$, defined from (8). Using this, we find that

$$\beta'_x = \operatorname{tr}(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} B^*\mathbb{X}^{-1}B) + \beta^2.$$

Then, using properties of the trace, it follows that

$$\beta''_{xx} = \operatorname{tr}(\begin{bmatrix} -2i & 0 \\ 0 & 0 \end{bmatrix} B^*\mathbb{X}^{-1}AB) + \operatorname{tr}(\begin{bmatrix} 0 & -2i\beta \\ 4i\beta & -2 \end{bmatrix} B^*\mathbb{X}^{-1}B) + 2\beta^3.$$

Finally,

$$\beta'''_{xxx} = \operatorname{tr}(\begin{bmatrix} -8i\beta & 4 \\ -4 & 0 \end{bmatrix} B^*\mathbb{X}^{-1}AB) + \operatorname{tr}(\begin{bmatrix} 0 & 4i\beta' - 4i\beta^2 \\ 0 & -4\beta \end{bmatrix} B^*\mathbb{X}^{-1}B) + 6(\beta')^2.$$

On the other hand, from the vessel conditions it follows that

$$\begin{aligned}
\beta'_t &= \operatorname{tr}(\begin{bmatrix} -2i\beta & 1 \\ -1 & 0 \end{bmatrix} B^*\mathbb{X}^{-1}AB) - \beta\operatorname{tr}(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B^*\mathbb{X}^{-1}B) + i(\beta' - \beta^2)\operatorname{tr}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B^*\mathbb{X}^{-1}B) \\
&\quad -i\beta\operatorname{tr}(\sigma_2 B^*\mathbb{X}^{-1}B\sigma_1 B^*\mathbb{X}^{-1}B).
\end{aligned}$$

Notice that the last term in this expression is zero by the normalization condition (24), since

$$\operatorname{tr}(\sigma_2 B^*\mathbb{X}^{-1}B\sigma_1 B^*\mathbb{X}^{-1}B) \quad = -\beta\operatorname{tr}(\sigma_1 B^*\mathbb{X}^{-1}B) = 0.$$

Comparing the last two expressions, we obtain the formula (26). □

3.1 The class of transfer functions

Before we define the class of the transfer functions of evolutionary KdV vessels $\mathfrak{V}_{KdV}$, we prove the following proposition:

Proposition 11. *Let $\mathfrak{E}$ be an evolutionary KdV vessel in Definition 8. Then its transfer function*

$$S(\lambda, x, t) = I - B^*(x,t)\mathbb{X}^{-1}(x,t)(\lambda I - A)^{-1}B(x,t)\sigma_1 \tag{27}$$

has the following properties:

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1. $S(\lambda, x, t)$ is defined on the set $\mathrm{D}_S = \mathrm{J} \times \mathrm{I}_x \times I_t$, where $\mathrm{I}_x, \mathrm{I}_t$ are intervals and $\mathrm{J} = \mathrm{spec}(A) \subseteq \mathbb{C}$,

2. For all $x \in \mathrm{I}_x, t \in \mathrm{I}_t$, $S(\lambda, x, t)$ is an analytic function of λ outside J, and $S(\infty, x, t) = I$, when λ approaches infinity, away from J,

3. For all $\lambda \notin \mathrm{J}$, $S(\lambda, x, t)$ is a differentiable function of x and of t,

4. $S(\lambda, x)$ satisfies the symmetry condition

$$S^*(-\bar{\lambda}, x, t)\sigma_1 S(\lambda, x, t) = \sigma_1 \qquad (28)$$

for all $(\lambda, x, t) \in \mathrm{D}_S$,

5. Multiplication by $S(\lambda, x, t)$ maps solutions $u(\lambda, x, t)$ of the **input** LDE with the spectral parameter λ:

$$-\sigma_1 \frac{\partial}{\partial x} u(\lambda, x, t) + (\sigma_2 \lambda + \gamma(x, t)) u(\lambda, x, t) = 0 \qquad (29)$$

to solutions $y(\lambda, x, t) = S(\lambda, x, t) u(\lambda, x, t)$ of the **output** LDE with the same spectral parameter:

$$-\sigma_1 (\frac{\partial}{\partial x} y(\lambda, x, t) + (\sigma_2 \lambda + \gamma_*(x, t)) y(\lambda, x, t) = 0. \qquad (30)$$

Proof: These properties are easily checked, and follow from formula (27) for $S(\lambda, x, t)$. The first and second properties are immediate. The third property follows from the differentiability assumptions on the operators $\mathbb{X}(x, t), B(x, t)$. The fourth property follows from straightforward calculations using the Lyapunov equation (6):

$$S^*(\mu, x, t)\sigma_1 S(\lambda, x, t) - \sigma_1 =$$
$$-(\bar{\mu} + \lambda)\sigma_1 B^*(x, t)(\bar{\mu} I - A^*)^{-1} \mathbb{X}^{-1}(x, t)(\lambda I - A)^{-1} B(x, t)\sigma_1 = 0$$

for $\mu = -\bar{\lambda}$. The fifth property follows directly from the definitions (by substituting $y(\lambda, x, t) = S(\lambda, x, t) u(\lambda, x, t)$ in (30) and using (29) for $u(\lambda, x, t)$), and from the formula (27) for $S(\lambda, x, t)$ (by using vessel conditions in order to differentiate it). $\square$

Definition 12. *The class $\mathcal{I}_{KdV}$ is defined as the class of functions $S(\lambda, x, t)$, defined on a set $\mathrm{D}_S = \mathrm{J} \times \mathrm{I}_x \times I_t$, where $\mathrm{I}_x, \mathrm{I}_t$ are intervals and $\mathrm{J} \subseteq \mathbb{C}$ is a closed subset, possessing the properties of the Proposition 11. The function is called **regular**, if it is analytic at infinity for all $x \in \mathrm{I}_x, t \in \mathrm{I}_t$.*

In the regular case (i.e. all operators are bounded and the transfer function is analytic at infinity), similar to [Melb, Theorem 2.7], we obtain the following realization theorem.

Theorem 13 (Realization in regular case). *Given a transfer function $S(\lambda, x) \in \mathcal{I}_{KdV}$, for which additionally the set J is bounded, there exists a regular KdV evolutionary vessel $\mathfrak{V}_{KdV}$ whose transfer function coincides with $S(\lambda, x, t)$, probably on smaller intervals $\mathrm{I}'_x \subseteq \mathrm{I}_x, \mathrm{I}'_t \subseteq \mathrm{I}_t$.*

Proof: Identical to [Melb, Theorem 2.7]. $\square$

Notice that, given a function $S(\lambda, x, t) \in \mathcal{I}_{KdV}$, we can take a function

$$H = \left[\begin{array}{cc} h_1(\lambda) & \dfrac{ih_3(\lambda)}{\lambda} \\ h_3(\lambda) & h_1(\lambda) \end{array} \right], \qquad (31)$$

which commutes with the fundamental matrix of the input LDE. As a result $S(\lambda, x, t)H \in \mathcal{I}_{KdV}$ and will correspond to the same vessel parameters (see [MVb] for the details). This turns out to be the only way of generating a function, corresponding to the same vessel parameters.

Theorem 14 (Uniqueness in regular case). *Given KdV parameters $\sigma_1, \sigma_2, \gamma, \gamma_*(x, t)$, there exists a unique initial value $S(\lambda, 0, 0)$ up to a symmetric, identity at infinity matrix function $H(\lambda)$, defined in (31).*

Proof: Identical to [MVb, Lemma 4.3]. $\square$

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4 Examples of solutions of the KdV equation

In this section we show different classes of solutions of the KdV equation (1) arising from different choices of evolutionary KdV vessels. In Table 1 we presented classes of solutions, arising from different types of evolutionary vessels. In the periodic and almost-periodic odd cases it is shown in the next sections that the correspondence is 1-1. Adjusting the classical inverse scattering theory, we can show that if a solution of the KdV equation satisfies at $t = 0$ the condition $\int_{\mathbb{R}}(1 + |x|)q(x)dx < \infty$ and is three times differentiable, it can be realized by a KdV vessel.

The last pages are devoted to a proof of the correspondence between classes of solutions of (1), appearing in column one and the corresponding evolutionary vessels, appearing in column two. We consider only the case of Dissipative operators $\mathbb{X}_0 = I$ and solutions of (1), which are odd in x. Moreover, we take the operator A to be of a very simple "diagonal" form.

4.1 Analytic–exponential class

We first make a simple observation, related to the regular vessels.

Theorem 15. *Suppose that we are given a regular KdV vessel on an interval* I. *Then* $\beta(x,t)$ *can be extended to an analytic function in the neighborhood of* x_0 *on the interval* I_x *uniformly in* $t \in I_t$.

Proof: Let us realize the transfer function $S(\lambda, t_2)$ for a fixed value of x_0, t_0:

$$S(\lambda, t_2^0) = I - B_0^* \mathbb{X}_0^{-1}(\lambda I - A)^{-1}B_0\sigma_1,$$

where all the operators are bounded. Let us define two fundamental matrices $\Phi(x,\lambda), \Psi(\lambda, t)$ by

$$\frac{\partial}{\partial x}\Phi(x,\lambda) = -(\lambda\Phi(x,\lambda)\sigma_2 + \Phi(x,\lambda)\gamma)\sigma_1^{-1}, \quad \Phi(0,\lambda) = I,$$

$$\frac{\partial}{\partial t}\Psi(t,\mu) = -i(\mu^2\Psi(x,\mu)\sigma_2 + \mu\Psi(t,\mu)\gamma)\sigma_1^{-1}, \quad \Psi(0,\mu) = I.$$

Then due to (5) and (22), using the Danford-Schwartz calculus [ND88]

$$B(x,t) = \oint \oint (\lambda I - A)^{-1}(\mu I - A)^{-1}B_0\Phi(x,\lambda)\Psi(t,\mu)d\lambda d\mu$$

for $R > \|A\|$. Since the function $\Phi(x,\lambda), \Psi(t,\lambda)$ satisfies LDEs with constant coefficients, it follows that $B(x,t)$ can be extended to an entire function in both variables x, t. Similarly, one obtains that $\mathbb{X}(x,t)$ is an entire function of two variables.

Then $\mathbb{X}(x,t)$ is invertible in a neighborhood of $(0,0)$, since it is invertible at $(x,t) = (0,0)$ and satisfies a differential equation. Finally, from the linkage condition (8) for $\gamma_*(x,t)$, it follows that $\gamma_*(x,t)$ (and $\beta(x,t)$) are analytic in the same neighborhood. $\square$

Let us consider now an analytic potential $\beta(x,t)$. We will construct a transfer function $S(\lambda, x, t)$ using its Taylor series in λ. The Taylor coefficients are called the *Markov moments* and we denote them by $H_n(x,t)$:

$$S(\lambda, x, t) = I - \sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}H_n(x,t), \quad H_n(x,t) = \begin{bmatrix} H_n^{11}(x,t) & H_n^{12}(x,t) \\ H_n^{21}(x,t) & H_n^{22}(x,t) \end{bmatrix}. \tag{32}$$

Then using the differential equation (12), and comparing the coefficients of $\dfrac{1}{\lambda^n}$ we can obtain that ([MVb, Section 5.6])

$$\begin{cases} H_{n+1}^{12} & = iH_n^{21} - \frac{\partial}{\partial x}H_n^{11} + \beta H_n^{11}, \\ H_{n+1}^{11} - H_{n+1}^{22} & = i(\frac{\partial}{\partial x}H_{n+1}^{12} - \beta H_{n+1}^{12}), \\ \frac{\partial}{\partial x}(H_{n+1}^{11} + H_{n+1}^{22}) & = -i(\beta_x' - \beta^2)H_{n+1}^{12} + \beta(H_{n+1}^{11} - H_{n+1}^{22}), \\ 2i\frac{\partial}{\partial x}H_{n+1}^{21} & = \frac{\partial^2}{\partial x^2}H_{n+1}^{11} - 2\beta\frac{\partial}{\partial x}H_{n+1}^{11}. \end{cases} \tag{33}$$

Rearranging the terms and performing simple algebraic manipulations, we can obtain that all the entries depend on H_n^{12} as follows:

$$H_n^{11} \qquad = \frac{i(H_n^{12})'}{2} - i\int_0^x \beta'(y)H_n^{12}(y)dy, \tag{34}$$

$$(H_n^{22})' \qquad = \frac{i}{2}(H_n^{12})'' - i\beta(H_n^{12})', \tag{35}$$

$$(H_n^{21})' \quad = \frac{1}{2}(H_n^{12})''' - \beta(H_n^{12})'' - \beta'(H_n^{12})' + (2\beta\beta' - \beta'')H_n^{12}. \tag{36}$$

Differentiating the first equation in (33) and substituting H_n^{21} from the fourth row, we obtain that

$$H_n^{12} = -\frac{(H_{n-1}^{11})'}{2} + \int_0^x \beta'(y)H_{n-1}^{11}(y)dy. \tag{37}$$

Define an operator T on the space of analytic functions

$$Tf(x) = 2\int_0^x \beta'(y)f(y)dy \tag{38}$$

and define the semi-group, generated by T

$$G'(x) = TG(x), \quad G(0) = I,$$

and denote $H_0^{12} = GK_0$, from (34) it follows that $H_0^{11} = \frac{i}{2}GK_0'$. Similarly, from (37) it follows that $H_1^{12} = -\frac{i}{2^2}GK_0''$ and from (34) $H_1^{11} = -\frac{i^2}{2^3}GK_0'''$. By induction we can show that

$$H_n^{12} = \frac{(-i)^n}{2^{2n}}GK_0^{(2n)}, \quad H_n^{12} = -\frac{(-i)^n}{2^{2n+1}}GK_0^{(2n+1)}.$$

It is important that the moments will grow sub-exponentially uniformly in x, t for the existence of the transfer function $S(\lambda, x, t)$ as the following theorem states.

Theorem 16. *An analytic on $I_x \times I_t$ function $\beta(x, t)$ can be realized by a KdV vessel, if and only if there exists $M, r > 0$ such that on $I_x \times I_t$ for $K(x, t) = G^{-1}\beta(x, t)$ it holds that*

$$|\frac{\partial^n}{\partial x^n}K(x, t)| \leq Mr^n. \tag{39}$$

Proof: The necessity of this condition follows from the preceding the theorem arguments, since the moments are of the form $H_n(x, t) = B(x, t)\mathbb{X}^{-1}A^n B(x, t)$ and

$$\|H_n\| \leq \|B(x, t)\|\|\mathbb{X}^{-1}\|\|A\|^n\|B(x, t)\| \leq Mr^n,$$

for $M' = \|B(x, t)\|\|\mathbb{X}^{-1}\|\|B(x, t)\|, r = \|A\|$. Therefore

$$|K(x, t)| = \|G^{-1}\|H_n^{12}\| \leq \|G^{-1}\|M'r^n = Mr^n, \quad M = \|G^{-1}\|M'.$$

The converse also holds, since if the condition (39) holds one can define a function

$$S(\lambda, x, t) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}H_n(x, t)$$

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which will be analytic at infinity $\lambda = \infty$, and satisfy the differential equation (12). In corollary 20 we show that such a function can always be "fixed" to a symmetric function. $\square$

Let us concentrate now on the symmetry, which was not taken into account while constructing the transfer function in Theorem 16. Notice that the symmetry condition (28) can be checked at $x = 0$ only, since the expression

$$S^*(-\bar\lambda, x, t)\sigma_1 S(\lambda, x, t)$$

has derivative zero with respect to x and t. This follows immediately from the vessels condition (5), (7), (6), (22), and (23). So, it is enough to require this condition at $(x, t) = (0, 0)$. Taking a matrix of the form $H(\lambda) = \begin{bmatrix} h_1(\lambda) & \dfrac{ih_3(\lambda)}{\lambda} \\ h_3(\lambda) & h_1(\lambda) \end{bmatrix}$ (31), we know from arguments preceeding Theorem 14 that $S(\lambda, x, t)H(\lambda)$ is in the class $\mathcal{I}_{KdV}$ if and only if $S(\lambda, x, t) \in \mathcal{I}_{KdV}$. Let us show that arbitrary matrix $K(\lambda)$ may be brought to a special form which in turn may be brought to a symmetric one. We use the notation $k^{\star}$ *for the reflection with respect to the imaginary axis*:

$$k^{\star} = k^*(-\bar\lambda).$$

Lemma 17. *For an arbitrary matrix* $K(\lambda) = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$, *there exist the following decomposition*

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} = \begin{bmatrix} s_1 & s_3 - d \\ s_3 & s_1 \end{bmatrix} \begin{bmatrix} h_1 & \dfrac{i}{\lambda}h_3 \\ h_3 & h_1 \end{bmatrix}, d = \dfrac{s_3 s_1^{\star} + s_1 s_3^{\star}}{s_1 + s_1^{\star}}.$$

Proof: Choose the following parameters:

$$F = \frac{k_1 - k_4}{k_2 - \dfrac{i}{\lambda}k_3}, \quad G = \frac{k_2 - k_4 F}{k_4 - k_2 \dfrac{i}{\lambda}F},$$

$$H = -\frac{k_1}{k_4 F}(1 + FG\frac{i}{\lambda}) + G + \frac{1}{F}, \quad T = \frac{k_1}{H^{\star}(1 + F(G - H))}$$

$$p^2 = \frac{TT^{\star}}{T + T^{\star}}(F + F^{\star})\frac{1}{HH^{\star}} \quad (p = p^{\star}),$$

Define next the entries as follows

$$h_1 = \frac{T}{p}, \quad h_3 = F h_1, \quad s_1 = H^{\star}p, \quad s_3 = G s_1.$$

Then by straightforward calculations we can show that the decomposition in the lemma holds. Moreover, we can choose k_1, k_4 so that the behavior at infinity of all these functions is as follows

$$F(\infty) = 0 \Rightarrow G(\infty) = 0 \Rightarrow H(\lambda) \approx -\frac{1}{F}\frac{-k_1 + k_4}{K_4} = \frac{k_2 - \dfrac{i}{\lambda}k_3}{k_4} \approx \frac{1}{\lambda}.$$

Then

$$T(\lambda) = \frac{k_1}{H^*(1 + F(G - H))} \approx \lambda$$

and

$$p^2(\lambda) \approx \frac{(\lambda^2)}{2\lambda}(\frac{2}{\lambda})\frac{1}{\dfrac{1}{\lambda^2}} = \lambda^2 \Rightarrow p(\lambda) \approx \lambda.$$

As a result

$$Hp \approx \frac{1}{\lambda}\lambda = const.$$

Consequently, s_1 is a constant at infinity and s_3 is zero. $\square$

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Lemma 18. *Suppose that $K(\lambda)$ is a constant at $\lambda = \infty$ and is of the form*

$$
\begin{bmatrix} s_1 & s_3 - d \\ s_3 & s_1 \end{bmatrix}, d = \frac{s_3 s_1^\star + s_1 s_3^\star}{s_1 + s_1^\star}.
$$

Then defining $h_1(\lambda), h_3(\lambda)$ such that

$$
\begin{cases} h_1 h_1^\star + \dfrac{i}{\lambda} h_3 h_3^\star = \dfrac{1}{1 + s_1 s_1^\star}, \\ h_1 h_3^\star + h_3 h_1^\star = 0, \end{cases}
$$

and the matrix $H = \begin{bmatrix} h_1(\lambda) & \dfrac{ih_3(\lambda)}{\lambda} \\ h_3(\lambda) & h_1(\lambda) \end{bmatrix}$, we will obtain that $K(\lambda)H$ is symmetric.

Proof: Immediate calculation, by substituting all the formulas into the symmetry condition

$$
K(\lambda)H(\lambda)\sigma_1 H^*(-\bar{\lambda})K^*(-\bar{\lambda}) = \sigma_1.
$$

$\square$

The condition (39) can be translated on a condition on $\beta(x,t)$. Indeed, from the definition of $K(x)$ it follows that $\beta(x,t) = G(x)K(x,t) = e^{Tx}K(x,t)$, where the operator T on the set of analytic functions on $I_x \times I_t$ is defined by (38). Consequently, using the Leibniz rule for the differentiation we obtain

$$
|\beta^{(n)}(x,t)| \le \sum_{i=0}^n \binom{n}{i} \|T^i e^{Tx}\| |K^{(n-i)}(x,t)| \le
$$

$$
\le \sum_{i=0}^n \binom{n}{i} \|T\|^i \|e^{Tx}\| M r^{n-i} = M(\|T\| + r)^n \|e^{Tx}\| \le ML^n e^{bx} \le M \frac{d^n}{dx^n} e^{kx},
$$

for $L = \|T\| + r, b = \|T\|$ and $k = \max(L, b)$.

Definition 19. *The class of analytic functions $\beta(x,t)$ on $I_x \times I_t$, for which there exists constants $M, k > 0$ such that*

$$
|\beta^{(n)}(x,t)| \le M \frac{d^n}{dx^n} e^{kx} \tag{40}
$$

*is denoted by $\boldsymbol{A}(k, I_x \times I_t)$ and is called **analytic–exponential** class.*

Corollary 20. *Every function $\beta(x,t) \in \boldsymbol{A}(k, I_x \times I_t)$ can be realized using a KdV evolutionary vessel.*

Proof: Construct (32) the transfer function $S(\lambda, x, t)$ using the moments $H_n(x,t)$ satisfying the conditions (33). Then by Theorem 16 it satisfies the differential equation (12) and using Lemma 18 it can be brought to a symmetric form. Thus $S(\lambda, x, t)$ belongs to the class $\boldsymbol{\mathcal{I}}_{KdV}$ and is regular (i.e. analytic in λ at infinity). By Theorem 13 $S(\lambda, x, t)$ can be realized as the transfer function of a KdV evolutionary vessel. $\square$

4.2 Continuous case: $\mathcal{H} = L^2(\Gamma)$

In the classical scattering theory [Fad63], under assumption $\int_0^\infty x|q(x)|dx < \infty$, it follows that the spectrum of the corresponding operator $-\dfrac{d^2}{dx^2} + q(x)$ consists of a finite number of points and a cut along the imaginary axis. Since we are dealing with the pure continuous spectrum, we will refer the potentials corresponding to $\Gamma \subseteq i\mathbb{R}^+$ as *classical potentials*.

Let us first delineate the construction of a vessel on the curve $\Gamma = i\mathbb{R}_+$. We define $\mathcal{H} = L^2(\Gamma)$ and the operators of the vessel as follows (using notation $\mu = is^2, \delta = ir^2$):

$$A \quad\quad = \mathrm{diag}(i\mu) = -A^*,$$

$$B(x,t) \quad\quad = b(\mu)\left[\begin{array}{cc} \dfrac{\sin(sx - s^3 t)}{s} & -i\cos(sx - s^3 t) \end{array}\right],$$

$$(\mathbb{X}(x,t)f)(\mu) \quad = f(\mu) + \int\limits_0^\infty \frac{\dfrac{\sin(sx - s^3 t)}{s}\cos(rx - r^3 t) - \cos(sx - s^3 t)\dfrac{\sin(rx - r^3 t)}{r}}{s^2 - r^2} b(\mu)b(\delta)^* f(\delta)dr$$

$$(41)$$

It is important to notice that the involved operators are well-defined. The functions $\cos(sx - s^3 t)$ and $\dfrac{\sin(sx - s^3 t)}{s}$ are actually functions of $s^2 = -i\mu$. The operator $\mathbb{X}(x,t)$ acts on functions $f(\mu) \in L^2(\Gamma)$ and gives as the output function $(\mathbb{X}(x,t)f)(\mu)$.

Lemma 21. *The collection $\mathfrak{E}$, defined by the operators in* (41) *is a KdV evolutionary vessel.*

Proof: Similar to the proof of Theorem 9. $\qquad\square$

Lemma 22. *The following equality holds for any $g(\mu) \in \mathcal{H}$*

$$\mathbb{X}(x,0)[g(\mu)\frac{\sin(sx)}{s}] = g(\mu)\frac{\sin(sx)}{s}.$$

Proof: Notice that the right hand side of the equality satisfies the following differential equation with initial conditions (recall that $\mu = is^2$):

$$\frac{\partial^2}{\partial x^2}Y(\mu, x) + i\mu Y(\mu, x) = 0, \quad Y(\mu, 0) = 0, \quad Y'(\mu, 0) = g(\mu).$$

Let us show that the left hand side satisfies the same equation and the same initial conditions. Since $\mathbb{X}(0) = I$ and $\mathbb{X}'(0) = B(0)\sigma_2 B(0)^* = 0$, we obtain that

$$\mathbb{X}(0)[g(\mu)\frac{\sin(s0)}{s}] = 0, \quad \frac{\partial}{\partial x}\mathbb{X}(x)[g(\mu)\frac{\sin(sx)}{s}]|_{x=0} = \mathbb{X}'(0)[g(\mu)\frac{\sin(s0)}{s}] + \mathbb{X}(0)g(\mu)\cos(s0) = g(\mu),$$

which means that $\mathbb{X}(x)[g(\mu)\frac{\sin(sx)}{s}]$ satisfies the same initial conditions. Then

$$\frac{\partial^2}{\partial x^2}\mathbb{X}(x)[g(\mu)\frac{\sin(sx)}{s}] + i\mu\mathbb{X}(x)[g(\mu)\frac{\sin(sx)}{s}] =$$

$$= \mathbb{X}''(x)[g(\mu)\frac{\sin(sx)}{s}] + 2\mathbb{X}'(x)[g(\mu)\cos(sx)] + \mathbb{X}(x)[g(\mu)s\sin(sx)] + i\mu\mathbb{X}(x)[g(\mu)\frac{\sin(sx)}{s}]$$

$$= \int\limits_0^L [B(\mu, x)\left[\begin{array}{cc} 0 & i \\ -i & 0 \end{array}\right]B^*(\delta, x)g(\delta)\frac{\sin(tx)}{t} + 2B(\mu, x)\sigma_2 B^*(\delta, x)g(\delta)\cos(tx)]dt$$

$$\quad + \int\limits_0^L [\frac{B(\mu, x)\sigma_1 B^*(\delta, x)}{\mu - \delta}[g(\delta)t\sin(tx) + g(\delta)i\mu\frac{\sin(tx)}{t}]dt$$

$$= \int\limits_0^L [B(\mu, x)\left[\begin{array}{cc} 0 & i \\ -i & 0 \end{array}\right]B^*(\delta, x)g(\delta)\frac{\sin(tx)}{t} + 2B(\mu, x)\sigma_2 B^*(\delta, x)g(\delta)\cos(tx)]dt$$

$$\quad + \int\limits_0^L [\frac{B(\mu, x)\sigma_1 B^*(\delta, x)}{\mu - \delta}g(\delta)i(\mu - \delta)\frac{\sin(tx)}{t}]dt$$

$$= \int\limits_0^L [B(\mu, x)\left[\begin{array}{cc} 0 & 2i \\ 0 & 0 \end{array}\right]B^*(\delta, x)g(\delta)\frac{\sin(tx)}{t} + 2B(\mu, x)\sigma_2 B^*(\delta, x)g(\delta)\cos(st)]dt$$

$$= \int\limits_0^L 2ic(\mu)\frac{\sin(sx)}{s}i\cos(tx)g(\delta)\frac{\sin(tx)}{t} + 2c(\mu)\frac{\sin(sx)}{s}\frac{\sin(tx)}{t}g(\delta)\cos(tx)dt = 0. \quad\square$$

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Corollary 23. *The formula for the potential is*

$$q(x) = 2 \int_0^L c(\mu) c^*(\mu) \frac{\sin(2sx)}{s} ds. \tag{42}$$

Proof: From the previous theorem, it follows that

$$\mathbb{X}^{-1}(x)[c(\mu)\frac{\sin(sx)}{s}] = c(\mu)\frac{\sin(sx)}{s}.$$

Then the result is immediate from formulas (20) and (21). $\qquad\square$

Theorem 24. *There is a 1-1 correspondence between potentials $q(x)$ and functions $c(\mu)$ satisfying (42).*

From the uniqueness of the solution of the KdV equation we have the following theorem.

Theorem 25. *There is 1-1 correspondence between $\beta(x,t)$ satisfying the KdV equation (1) and evolutionary KdV vessels, defined on the curve $\Gamma = i\mathbb{R}_+$ using $c(\mu)$ defined from (42).*

4.3 Discrete case: $\mathcal{H} = \ell^2$

In this section we present solutions for which the inner space $\mathcal{H}$ is discrete. Suppose that the space $\mathcal{H} = \ell^2$ consists of infinite column-sequences $((\cdot)^t$-denotes the transpose):

$$\mathcal{H} = \ell^2 = \{\bar{\mathbf{x}}_n^t = (x_1, x_2, \ldots)^t \mid \sum_{n=1}^\infty |x_n|^2 < \infty\}.$$

Let $\bar{\mathbf{k}}_n = (k_1, k_2, \ldots)$ be a sequence of mutually different numbers ($k_i \neq k_j$ for $i \neq j$) and $\bar{\mathbf{b}}_n = (b_1, b_2, \ldots) \in \ell^2$ be another sequence. We define analogously to the finite dimensional case (25) the following operators:

$$
\begin{aligned}
A_p &= \operatorname{diag}(ik_n^2) = -A^*, \\
B_p(x,t) &= \operatorname{diag}(b_n)\left[\begin{array}{cc} \dfrac{\sin(k_n x - k_n^3 t)}{k_n} & -i\cos(k_n x - k_n^3 t) \end{array} \right], \\
\mathbb{X}_p(x,t) &= I + [\dfrac{\dfrac{\sin(k_n x - k_n^3 t)}{k_n}\cos(k_m x - k_m^3 t) - \cos(k_n x - k_n^3 t)\dfrac{\sin(k_m x - k_m^3 t)}{k_m}}{k_n^2 - k_m^2} b_n b_m^*]
\end{aligned} \tag{43}
$$

and the collection realizing an odd (for $t = 0$) periodic solution of the KdV equation

$$\mathfrak{E}_{p,odd} = (A_p, B_p(x,t), \mathbb{X}_p(x,t); \sigma_1, \sigma_2, \gamma, \gamma_*(x,t); \mathcal{H}, \mathbb{C}^2)$$

Lemma 26. *The collection $\mathfrak{E}_{p,odd}$, defined by the operators in (43) is a KdV evolutionary vessel.*

Proof: Similar to the proof of Theorem 9. $\qquad\square$

In order to reconstruct a vessel for a given potential, we first notice that for $t = 0$ it is similar to the classical inverse scattering problem. As in the continuous case $\mathcal{H} = L^2(\Gamma)$, we can calculate the inverse $\mathbb{X}(x,0)$.

Lemma 27. *The following equality holds*

$$\mathbb{X}(x)[b_n\frac{\sin(k_n x)}{k_n}] = b_n\frac{\sin(k_n x)}{k_n}.$$

Proof: Since $\mathbb{X}(0,t) = I$ and both functions satisfy the same differential equation, the lemma follows. $\quad\square$

As a result, we have the following corolalry:

Corollary 28. *The formula for $\beta(x,0)$ is*

$$\beta(x,0) = \sum_n b_n b_n^* \frac{\sin^2(k_n x)}{k_n^2}. \tag{44}$$

Recall that in the KdV equation (26) one uses the third derivative of β. As a result, we must demand that the third derivative of $\sum_n b_n b_n^* \frac{\sin^2(k_n x)}{k_n^2}$ exists. Differentiating this expression three times, we obtain that the following sum must converge:

$$\sum_n b_n b_n^* k_n \sin(2k_n x).$$

Therefore the necessary and obviously sufficient condition for the existence of the vessel is that the sequence $b_n \sqrt{k_n}$ is square summable.

Finally, we can use the uniqueness of the solution of the KdV equation in order to obtain the following theorem.

Theorem 29. *There is a 1-1 correspondence between KdV evolutionary vessels $\mathfrak{V}_{KdV,p}$ and solutions $\beta(x,t)$ of (26). The correspondence is implemented by the formula (44).*

Proof: For $t = 0$, we will obtain that $\beta(x,0)$ is represented by the formula (44). Let us construct a KdV evolutionary vessel $\mathfrak{V}_{KdV}$ using $\mathcal{H} = \ell^2$ and the sequence $\{b_n\}$, obtained in from the Fourier-sinus transform of $\beta(x,0)$ (44). Then the $\beta_{KdV}(x,t)$ of the vessel $\mathfrak{V}_{KdV,p}$ satisfies the same KdV equation (1) as the given $\beta(x,t)$. By the uniqueness of the solution they coincide. In other words, the vessel $\mathfrak{V}_{KdV}$ realizes $\beta(x,t)$. $\qquad\qquad\square$

Notice that following the same lines, we could obtain a KdV vessel, realizing an almost periodic, odd solution of (1). For this we need to use the Bohr spectrum of the function $\beta(0,t)$. In this cacse the formula (44) becomes

$$\beta(x,0) = \sum_n b_n b_n^* \frac{\sin^2(r_n x)}{r_n^2}. \tag{45}$$

for $b_n \sqrt{k_n}$ square summable, and $r_n \in \mathbb{R}$. A KdV vessel $\mathfrak{V}_{KdV,ap}$ realizing an almost periodic solution is identical to the periodic case with k_n's substituted by r_n's. The analogue of Theorem 29 is as follows.

Theorem 30. *There is a 1-1 correspondence between KdV evolutionary vessels $\mathfrak{V}_{KdV,ap}$ and almost periodic odd solutions $\beta(x,t)$ of (26). The correspondence is implemented by the formula (45).*

4.4 Explicit formula for $\beta(x,t)$ when $\mathcal{H} = \ell^2$

In order to explicitly construct a solution of the KdV equation for $\beta(x,t)$ (26) we suppose that

$$\beta(x,t) = \sum_n |b_n(t)|^2 \frac{\sin^2(k_n x - k_n^3 t)}{k_n^2}.$$

Then substituting it into equation (26), we will obtain the following equality

$$4\sum_n \frac{|b_n(t)|^2}{dt} \frac{\sin^2(k_n x - k_n^3 t)}{k_n^2} = -6(\beta_x'(x,t))^2.$$

In other words,

$$4\sum_n \frac{d|b_n(t)|^2}{dt} \frac{\sin^2(k_n x - k_n^3 t)}{k_n^2} = -6\sum_n \sum_m \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \sin(2k_n x - 2k_n^3 t)\sin(2k_m x - 2k_m^3 t)$$

$$= -6\sum_n \sum_m \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \frac{\cos(2(k_n - k_m)x - 2(k_n^3 - k_m^3)t) - \cos(2(k_n + k_m)x - 2(k_n^3 + k_m^3)t)}{2}. \tag{46}$$

Notice that
$$k_n^3 \pm k_m^3 = (k_n \pm k_m)^3 \mp 3k_n k_m (k_n \pm k_m).$$
Using the formula for $\cos(x+y)$ equation (46) becomes

$$
4\sum_n \frac{d|b_n(t)|^2}{dt} \frac{\sin^2(k_n x - k_n^3 t)}{k_n^2} \quad = -3\sum_n \sum_m \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \Big[
$$
$$
\cos\big(2(k_n - k_m)x - 2(k_n - k_m)^3 t\big) \cos\big(6k_n k_m(k_n - k_m)t\big)
$$
$$
- \sin\big(2(k_n - k_m)x - 2(k_n - k_m)^3 t\big) \sin\big(6k_n k_m(k_n - k_m)t\big)
$$
$$
- \cos\big(2(k_n + k_m)x - 2(k_n + k_m)^3 t\big) \cos\big(6k_n k_m(k_n + k_m)t\big)
$$
$$
+ \sin\big(2(k_n + k_m)x - 2(k_n + k_m)^3 t\big) \sin\big(6k_n k_m(k_n + k_m)t\big)\Big].
$$

Assume next that $-k_m = k_{-m}$ (i.e. that there is an antisymmetry condition on the numbers k_m). Then

$$
\sum_n \sum_m \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \big[\sin\big(2(k_n - k_m)x - 2(k_n - k_m)^3 t\big) \sin\big(6k_n k_m(k_n + k_m)\big)\big]
$$
$$
= \sum_n \sum_{-m} \frac{|b_n(t)b_{-m}(t)|^2}{k_n k_{-m}} \big[\sin\big(2(k_n - k_{-m})x - 2(k_n - -k_m)^3 t\big) \sin\big(6k_n k_{-m}(k_n - k_{-m})\big)\big]
$$
$$
= \sum_n \sum_m \frac{|b_n(t)b_{-m}(t)|^2}{-k_n k_m} \big[\sin\big(2(k_n + k_m)x - 2(k_n + k_m)^3 t\big) \sin\big(-6k_n k_m(k_n + k_m)\big)\big]
$$
$$
= \sum_n \sum_m \frac{|b_n(t)b_{-m}(t)|^2}{-k_n k_m} \big[\sin\big(2(k_n + k_m)x - 2(k_n + k_m)^3 t\big) \sin\big(6k_n k_m(k_n + k_m)\big)\big].
$$

Therefore, if $|b_{-m}| = |b_m|$, we obtain that the terms involving sin are canceled. As a result we obtain in the last formula that

$$
4\sum_n \frac{d|b_n(t)|^2}{dt} \frac{\sin^2(k_n x - k_n^3 t)}{k_n^2}
$$
$$
= -3\sum_n \sum_m \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \cos\big(2(k_n + k_m)x - 2(k_n + k_m)^3 t\big) \cos\big(6k_n k_m(k_n + k_m)t\big).
$$

Finally, it remains to demand that the set $\Gamma = \{k_n\}$ has the property $\Gamma + \Gamma = \Gamma$. We shall obtain that

$$
4\sum_N \frac{d|b_N(t)|^2}{dt} \frac{1 - \cos(2k_N x - 2k_N^3 t)}{2k_N^2} = -3 \sum_{k_n + k_m = k_N} \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \cos\big(6k_n k_m k_N t\big) \cos\big(2k_N x - 2k_N^3 t\big).
$$

In other words, $b_N(t)$ must satisfy the differential equation

$$
\frac{1}{k_N^2} \frac{d|b_N(t)|^2}{dt} = -\frac{3}{2} \sum_{k_n + k_m = k_N} \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \cos\big(6k_n k_m k_N t\big) \tag{47}
$$

and the normalization condition

$$
\sum_N \frac{d|b_N(t)|^2}{dt} \frac{1}{k_N^2} = 0. \tag{48}
$$

Notice that the condition $b_{-N}(t) = b_N(t)$ holds in this case, since

$$
\frac{1}{k_{-N}^2} \frac{d|b_{-N}(t)|^2}{dt} = -\frac{3}{2} \sum_{k_n + k_m = k_{-N}} \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \cos\big(6k_n k_m k_{-N} t\big)
$$
$$
= -\frac{3}{2} \sum_{k_{-n} + k_{-m} = k_N} \frac{|b_{-n}(t)b_{-m}(t)|^2}{k_{-n} k_{-m}} \cos\big(6k_{-n} k_{-m} k_N t\big)
$$
$$
= -\frac{3}{2} \sum_{k_n + k_m = k_N} \frac{|b_n(t)b_m(t)|^2}{k_n k_m} \cos\big(6k_n k_m k_N t\big)
$$
$$
= \frac{1}{k_N^2} \frac{d|b_N(t)|^2}{dt}.
$$

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The equality follows from the initial condition $b_N(0) = b_{-N}(0)$, which follows in turn from the oddness of $\beta(x,0)$. Thus we obtain the following theorem.

Theorem 31. *Let $\beta(x,t)$ be a solution of the KdV equation, possessing the property that $\beta(x,0)$ is an odd function, such that (44) holds for a set $\Gamma = \{k_n\}$. Assume that $-\Gamma = \Gamma$ (i.e. $-k_N = k_{-N}$) and $\Gamma + \Gamma = \Gamma$. Then,*

$$\beta(x,t) = \sum_N |b_N(t)|^2 \frac{\sin^2(k_N x - k_N^3 t)}{k_N^2}$$

is a solution of the integrated KdV equation (26) *if and only if $b_N(t)$ satisfies* (47) *and the normalization condition* (48).

5 Conclusions and Remarks

One can study a wide range of questions using evolutionary vessels and their generalizations. We list several questions that we believe have a great potential to contribute to the study of PDEs, corresponding to integrable systems.

1. Which PDEs are obtained from the SL equation by changing the evolutionary conditions (22) and (23)?

2. Can we develop a similar theory starting from Non-Linear Shrödinger Equation (NLS) [MVb]? The formulas developed in this work will look different, but the ideas will remain the same. The NLS vessels parameters are presented in the next definition.

   Definition 32. *Non-Linear Shrödinger (NLS) Equation parameters are given by*

   $$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
   $$\gamma_*(x,t) = \begin{bmatrix} 0 & \beta(x,t) \\ -\beta^*(x,t) & 0 \end{bmatrix}$$

3. How are the Segal-Wilson [GS85] loop groups solutions of the KdV equation obtained using vessels? What is the space $\mathcal{H}$ in this case?

4. There exist Krichever [Kri77] solutions of the KdV. How can they be implemented using vessels on curves [Melb]?

5. Can one realize the Zakarov-Shabat scheme by generalizing these types of vessels to infinite number of variables?

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